

MAS317 Linear Algebra II, 2007

Solutions to Coursework 1

1. The *transpose* A^T of an $m \times n$ matrix A is the $n \times m$ matrix given by $(A^T)_{ij} = A_{ji}$ for all i, j . [2 marks]

A is *congruent* to B if there exists an invertible matrix P over the field such that $B = P^T A P$. [3 marks]

(i) If A is congruent to B and B is congruent to C then there exist invertible matrices P and Q such that $B = P^T A P$ and $C = Q^T B Q$. [2 marks]

So $C = Q^T P^T A P Q = (PQ)^T A (PQ)$ by (1.8)(ii). [4 marks]

PQ is invertible as P and Q are. So A is congruent to C . [2 marks]

(ii) If A is congruent to B then there exists an invertible matrix P such that $B = P^T A P$. [1 marks]

Now $A = (P^T)^{-1} B P^{-1} = (P^{-1})^T B P^{-1}$, by (1.8)(iv). [4 marks]

Also P^{-1} is invertible and so B is congruent to A . [2 marks]

(iii) If A is congruent to a diagonal matrix D then D is congruent to A by (b) and so there exists an invertible matrix P such that $A = P^T D P$. [4 marks]

Now $A^T = (P^T D P)^T = P^T D^T (P^T)^T$ by (1.8)(ii) [2 marks]

$= P^T D^T P$ by (1.8)(iii), [2 marks]

Now, as D is diagonal, $D^T = D$, and so $A^T = P^T D P = A$, that is, A is symmetric. [2 marks]

2. $A =$ [12 marks]

$$\begin{pmatrix} 1 & 3 & -2 \\ 3 & 0 & 3 \\ -2 & 3 & 7 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + 2R_1]{R_2 \rightarrow R_2 - 3R_1} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -9 & 9 \\ 0 & 9 & 3 \end{pmatrix} \xrightarrow[C_3 \rightarrow C_3 + 2C_1]{C_2 \rightarrow C_2 - 3C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9 & 9 \\ 0 & 9 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9 & 9 \\ 0 & 0 & 12 \end{pmatrix} \xrightarrow{C_3 \rightarrow C_3 + C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 12 \end{pmatrix} = D. \quad [6 \text{ marks}]$$

To find P start with I_3 and just do the col operations above.

[3 marks]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} C_2 \rightarrow C_2 - 3C_1 \\ C_3 \rightarrow C_3 + 2C_1 \end{matrix} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad [6 \text{ marks}]$$

$$C_3 \rightarrow C_3 + C_2 \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = P. \quad [3 \text{ marks}]$$

3. The easiest way to deal with this is to start with row and corresponding column operations that will put a non-zero entry in the 1,1 position, and then proceed as before. For example,

$$B = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} \xrightarrow{C_1 \rightarrow C_1 + C_2} \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \quad [10 \text{ marks}]$$

$$\begin{matrix} R_2 \rightarrow R_2 - (1/2)R_1 \\ R_3 \rightarrow R_3 - (1/4)R_1 \end{matrix} \begin{pmatrix} 4 & 2 & 1 \\ 0 & -1 & -3/2 \\ 0 & -3/2 & -1/4 \end{pmatrix} \begin{matrix} C_2 \rightarrow C_2 - (1/2)C_1 \\ C_3 \rightarrow C_3 - (1/4)C_1 \end{matrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & -3/2 \\ 0 & -3/2 & -1/4 \end{pmatrix} \quad [12 \text{ marks}]$$

$$R_3 \rightarrow R_3 - (3/2)R_2 \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & -3/2 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{C_3 \rightarrow C_3 - (3/2)C_2} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D. \quad [6 \text{ marks}]$$

To find P start with I_3 and just do the col operations above.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1 \rightarrow C_1 + C_2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad [3 \text{ marks}]$$

$$\begin{matrix} C_2 \rightarrow C_2 - (1/2)C_1 \\ C_3 \rightarrow C_3 - (1/4)C_1 \end{matrix} \begin{pmatrix} 1 & -1/2 & -1/4 \\ 1 & 1/2 & -1/4 \\ 0 & 0 & 1 \end{pmatrix} \quad [6 \text{ marks}]$$

$$C_3 \rightarrow C_3 - (3/2)C_2 \begin{pmatrix} 1 & -1/2 & 1/2 \\ 1 & 1/2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = P. \quad [3 \text{ marks}]$$